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Majorana zero modes

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Abstract: We derive the zero mode solutions for a Majorana fermion in the background of a cosmic string and contrast it with the zero mode solution for a (neutral) Dirac fermion. A Majorana zero mode carries no vector or axial charge, and it cannot be bosonised. We study the implications for vorton formation and stability. In the massless limit stability of the zero mode is guaranteed by energy-momentum conservation. However, zero modes obtain an effective mass on string loops. It is found that the conditions under which current formation can be effective are exactly those for which zero mode decay is most likely to occur.

Keywords: Solitons Monopoles and Instantons, Supersymmetry Phenomenology, Cosmology of Theories beyond the SM, Beyond Standard Model.

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1. Introduction

The existence of Majorana mass terms is particularly important for particle-physics phenomenology in the context of neutrino physics [1]. Indeed, the see-saw mechanism [2], which can naturally explain the smallness of the observed left-handed neutrino masses [3], relies on the existence of a Majorana mass term for a standard model singlet, the right-handed neutrino. Right-handed neutrinos and the see-saw mechanism are a prediction of unified theories which are left-right symmetric, such as Patti-Salam and SO(10) grand unified theories (GUTs). After breaking of the gauged $\mathrm{U}(1)_{B-L}$ symmetry a Majorana mass for the the right-handed neutrino is generated through the Higgs mechanism.

In the context of cosmology, GUTs predict the existence of cosmic strings [4, 5]. If the Higgs field which form the string breaks $\mathrm{U}(1)_{B-L}$, $B-L$ cosmic strings are produced [6, 7]. They can be important for baryogenesis [6, 8]. Cosmic strings can also contribute to the cosmic microwave background anisotropies and they produce gravitational waves [9]. Further, fermions coupling to the string forming Higgs field can be trapped on the string, travelling along the string at the speed of light [10]. Such fermionic zero modes can give rise to superconducting strings [11]. Conserved fermionic currents can stabilise cosmic string loops against gravitational collapse, resulting in the formation of vortons which are potentially cosmologically catastrophic [12]. It was shown, however, that if there are fermionic zero modes which travel in opposite directions on the string they could scatter with each other, thereby strongly reducing the current [13, 14]. Hence, string loops with chiral currents are the best vorton candidates. In an anomaly free theory, chiral zero modes have to be neutral; they can either be Dirac or Majorana.

In this paper we will be interested in chiral zero modes and in particular in zero modes for Majorana fermions. Such zero modes arise naturally in $B-L$ cosmic strings, as the $\mathrm{U}(1)_{B-L}$ breaking Higgs field gives a Majorana mass to the right-handed neutrino [6]. They can produce the observed lepton asymmetry of the universe, independently of the reheating temperature after inflation [6, 7]. We will discuss the difference between Majorana and Dirac zero modes. Neutral Dirac currents are persistent by virtue of a conserved quantum number, whereas Majorana currents are persistent for kinematical reasons. The difference between the two currents shows up when the zero-mode obtains an effective mass. Decay and annihilation of Dirac zero modes can be suppressed or forbidden by virtue of lepton number conservation; no such suppression exists for Majorana fermions. We apply our results to vortons. Neutrinos zero modes were discussed before in a slightly different context [15].

This paper is organised as follows. In the next section we will give a brief introduction to Majorana and Dirac spinors. In section 3 we present the zero mode solutions for both the Dirac and Majorana spinors in the presence of a Nielsen-Olesen string. We quantise the string in section 3.2, and compute the conserved currents. Moreover, a description of the effective theory in 2D is given. We apply our results to vortons in section 4. Both current condensation and stability is discussed, highlighting the difference between Dirac and Majorana currents. We end with some concluding remarks.

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2. Dirac and Majorana spinors

In the early Universe when particles are still massless, there is no distinction between Dirac and Majorana fermions. All fermions and anti-fermions can be described by two-component Weyl spinors. In grand unified theories fermions and anti-fermions are in the same representation and therefore must be described by spinors with the same chirality. By convention, we use left-handed Weyl spinors.[1] Weyl spinors in 4-dimensions have two on-shell degrees of freedom.

Fermions get masses via the Higgs mechanism. The mass term involves a coupling between a fermion bilinear and a scalar field ϕ which acquires a non-vanishing vacuum expectation value (VEV). The mass term must be invariant under all symmetries of nature, and in particular, it must be invariant under Lorentz transformations.

Given two left-handed Weyl spinors ψ_L and χ_L, the Lorentz invariant bilinear which can be constructed is of the form

$$\chi_L^T \tilde{C} \psi_L + \overline{\chi_L} \tilde{C} \overline{\psi_L}^T \,. \tag{2.1}$$

If the Weyl spinors transform under some symmetry, the above term is in addition gauge invariant if ψ_L and χ_L are charge conjugate of each other, i.e., if $\chi_L = \psi_L^c$. The bilinear term given in eq. (2.1) then becomes:

$$(\psi_L^c)^T \tilde{C} \psi_L + \text{h.c.} = (\tilde{C} \overline{\psi_R}^T)^T \tilde{C} \psi_L + \text{h.c.} = \overline{\chi_R} \tilde{C}^T \tilde{C} \chi_L + \text{h.c.} = \overline{\psi_R} \psi_L + \overline{\psi_L} \psi_R \,. \tag{2.2}$$

This leads to the standard Dirac mass term

$$\mathcal{L}_D = -\overline{\psi_R} m_D \psi_L - \overline{\psi_L} m_D^\dagger \psi_R \tag{2.3}$$

with $m_D = \lambda\phi$ and λ a Yukawa coupling. The two Weyl spinors ψ_L and $\psi_R = \tilde{C} \overline{\psi_L^c}^T$ combine to make a Dirac spinor

$$\psi_D = \psi_L + \psi_R \,, \tag{2.4}$$

and if m_D is real, the mass term can be written as $\mathcal{L}_D = -m_D \overline{\psi_D} \psi_D$. A Dirac spinor has on shell four real degrees of freedom corresponding to particle/antiparticle and positive/negative helicities. The Dirac mass term is invariant under a global U(1) transformation; this implies the existence of a quantum number (such as lepton number) which enables to distinguish particles from antiparticles. A Dirac mass term can be written for both neutral and charged particles.

Now starting with a single neutral Weyl spinor, i.e., a particle which does not carry any quantum number under any unbroken symmetry, we can also construct, by setting $\chi_L = \psi_L$ in eq. (2.1), the following Lorentz invariant:

$$\psi_L^T \tilde{C} \psi_L + \overline{\psi_L} \tilde{C} \overline{\psi_L}^T = \overline{\psi_R^c} \psi_L + \overline{\psi_L} \psi_R^c \,. \tag{2.5}$$

We can then write down what is called a Majorana mass term

$$\mathcal{L}_M = -\frac{1}{2} \overline{\psi_R^c} m_M \psi_L - \frac{1}{2} \overline{\psi_L} m_M^\dagger \psi_R^c \,, \tag{2.6}$$

[1] Our notation and conventions are detailed in the appendix.

Dirac	ϕ	ψ_L	χ_L
X	r	n	$p=-n-r$
Q	0	q	$-q$

Majorana	ϕ	ψ_L
X	r	$-r/2$
Q	0	0

Table 1: Charges of the Higgs field ϕ and of the fermion fields ψ_L and χ_L, which acquire respectively Dirac and Majorana mass by coupling with ϕ, under the broken $\mathrm{U}(1)_X$ and any unbroken $\mathrm{U}(1)_Q$. Here $r, n \neq 0$, but q can either be zero or non-zero.

where $m_M = \lambda\phi$ and the factor half is conventional. Analogously to the Dirac spinor, a 4-component Majorana spinor can then be defined

$$\psi_M = \psi_L + \tilde{C}\overline{\psi_L^c}^T = \psi_L + \psi_R^c = \psi_{M,L} + \psi_{M,R}\,. \tag{2.7}$$

If m_M is real, the mass term simplifies to $\mathcal{L}_M = -\frac{1}{2}m_M\overline{\psi_M}\psi_M$. A Majorana mass term is not invariant under a global U(1) transformation, and there is no quantum number which allows to distinguish particles from antiparticles (for instance, lepton number cannot be defined for a Majorana neutrino). A Majorana spinor is made of a single Weyl spinor and it has on shell two real degrees of freedom corresponding to the two helicity states. It satisfies the Majorana condition

$$\psi_M^c = \psi_M\,, \tag{2.8}$$

that is, a Majorana particle is its own antiparticle.

Majorana masses are particularly important in understanding the smallness of the neutrino masses via the see-saw mechanism [2]. Indeed, the latter relies on the existence of a superheavy Majorana mass term for the right-handed neutrino on top of a standard Dirac mass term which arises from the coupling of both left and right-handed neutrinos to the standard model Higgs. In principle there can also be a small Majorana mass for the left-handed neutrino. There are then two distinct mass eigenstates which are both Majorana: a state with a tiny mass which is a superposition of mostly left-handed neutrino with a small admixture of the right-handed neutrino, and a superheavy state which is mostly composed of the right-handed neutrino. In the context of topological defects, we will be interested in a Majorana mass for the right-handed neutrino which arises as a consequence of gauge symmetry breaking.

In summary, massive charged fermions are always Dirac fermions whereas neutral fermions can either be Dirac or Majorana. Which mass terms can be constructed depends on the quantum numbers of the fermions and the scalar fields of the model. To be explicit, consider a $\mathrm{U}(1)_X \times \mathrm{U}(1)_Q$ gauge symmetry, possibly embedded in a larger gauge group. The $\mathrm{U}(1)_X$ symmetry is broken by the non-vanishing vacuum expectation value of a complex Higgs field ϕ, while $\mathrm{U}(1)_Q$ remains unbroken. Given two left-handed Weyl fermions ψ_L and χ_L, Dirac and Majorana mass terms involving the bilinear term given in eq. (2.1) multiplied by the Higgs field ϕ can be constructed if the charges of the fields are as given in table 1. Let's call X and Q the charges of the fields under $\mathrm{U}(1)_X$ and $\mathrm{U}(1)_Q$ respectively. ϕ breaks $\mathrm{U}(1)_X$ and keeps $\mathrm{U}(1)_Q$ unbroken, hence $X(\phi) = r \neq 0$ and $Q(\phi) = 0$. Let $X(\psi_L) = n$ and $X(\chi_L) = p$. Then the Yukawa term is gauge invariant only if $n+p+r=0$ and $Q(\psi_L) = -Q(\chi_L)$. We distinguish three cases:

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1. If $Q(\psi_L) = -Q(\chi_L) \neq 0$, we have a Dirac mass term. The spinors ψ_L and χ_L combine to form a Dirac spinor which has four on shell degrees of freedom. It describes a charged fermion.

2. If $Q(\psi_L) = 0$ and $n = -r/2$, we have a Majorana mass. A Majorana spinor is formed out of ψ_L, it has two on-shell degrees of freedom. It describes a Majorana fermion, a particle which is its own antiparticle.

3. Now if $Q(\psi_L) = 0$ and $n \neq r/2$, we have a Dirac mass term for a neutral particle. In this case, ψ_L and χ_L combine to form a Dirac spinor. Note that in this case always a global charge can be defined which distinguishes particles from antiparticles. If there is a second Higgs field with charge $X = -2X(\psi)$, respectively $X = -2X(\chi)$, a Majorana mass term can be constructed for ψ (respectively χ). There are then two distinct mass eigenstates which are Majorana.

3. Chiral currents

3.1 Zero modes solution

In this section, we write down the fermionic equations of motion in the background of a cosmic string. We give the zero mode solutions for both Dirac and Majorana fermions, and compare the results. We only discuss abelian cosmic strings; our results can be generalised straightforwardly to non-abelian strings.

Consider a $\mathrm{U}(1)_X$ gauge symmetry which is broken down to the identity by a Higgs field ϕ, when the latter acquires a non-vanishing VEV. Since $\pi_1(\mathrm{U}(1)_X) \neq 0$, cosmic strings form at the $\mathrm{U}(1)_X$ breaking scale according to the Kibble mechanism [4]. For a straight infinite cosmic string lying along the z-axis, the Higgs field ϕ and gauge field A have the form

$$\phi = \phi_0 f(r) e^{in\theta}, \tag{3.1}$$

$$A_\theta = -n\frac{g(r)}{er}, \qquad A_z = A_r = 0\,, \tag{3.2}$$

with ϕ_0 the Higgs VEV in the vacuum, and $n \in N$ the string's winding number. In the following we will concentrate on the minimum energy configuration corresponding to $|n| = 1$. The profile functions $f(r)$ and $g(r)$ must satisfy the following boundary conditions

$$f(0) = g(0) = 0, \tag{3.3}$$

$$\lim_{r\to\infty} f(r) = \lim_{r\to\infty} g(r) = 1\,. \tag{3.4}$$

The exact form of the functions $f(r)$ and $g(r)$ depends on the Higgs potential [9], which we do not specify.

We first discuss the well known zero mode solution [10] for a Dirac fermion getting its mass by coupling to ϕ^* [11]. The lagrangian is

$$\mathcal{L}_D = \overline{\psi_L} i\gamma^\mu D_\mu \psi_L + \overline{\psi_R} i\gamma^\mu D_\mu \psi_R - \lambda\phi^* \overline{\psi_R}\psi_L - \lambda\phi\overline{\psi_L}\psi_R\,, \tag{3.5}$$

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with $D_\mu = \partial_\mu + iqA_\mu$, λ the Yukawa coupling, and q the charges of $\psi_{L,R}$ under $\mathrm{U}(1)_X$. Note that $q_\phi - q_L + q_R = 0$; in general q_L and $-q_R$ can be different. The equations of motions are:

$$\begin{aligned} i\gamma^\mu D_\mu \psi_L &= \lambda\phi\psi_R, \\ i\gamma^\mu D_\mu \psi_R &= \lambda\phi^*\psi_L \,. \end{aligned} \tag{3.6}$$

From Jackiw and Rossi [10], and an index theorem [16], we know that there is one normalisable zero mode solution ψ^0 (a static solution) which is z and t independent. The zero mode solution are eigenvectors of the projection operator [11]

$$\gamma^0\gamma^3\psi^0 = n\psi^0 \tag{3.7}$$

with $n = 1$ for a vortex and $n = -1$ for an anti-vortex. With our choice of gamma matrices (see the appendix) the zero mode solutions are

$$\psi^0_{n=+1} = \begin{pmatrix} 0 \\ \alpha_+(r,\theta) \\ \beta_+(r,\theta) \\ 0 \end{pmatrix}, \qquad \psi^0_{n=-1} = \begin{pmatrix} \alpha_-(r,\theta) \\ 0 \\ 0 \\ \beta_-(r,\theta) \end{pmatrix}, \tag{3.8}$$

for a vortex and an anti-vortex respectively. The wave functions $\alpha_\pm(r,\theta)$ and $\beta_\pm(r,\theta)$ are of the form $f(r)e^{il\theta}$ with l an integer and $f(r)$ a real function localised on the string. We define

$$\begin{aligned} \alpha_\pm &= \int r\mathrm{d}r\mathrm{d}\theta\, \alpha_\pm(r,\theta)\,, \\ \beta_\pm &= \int r\mathrm{d}r\mathrm{d}\theta\, \beta_\pm(r,\theta)\,. \end{aligned} \tag{3.9}$$

The ratio $\alpha_\pm/\beta_\pm$ is determined by the equations of motion, more specifically, by the charges q_L and q_R. Further the sum $(\alpha_\pm^2 + \beta_\pm^2)$ is set by the spinor normalisation. Hence there is no freedom in choosing the (relative) magnitudes of $\alpha_\pm$ and $\beta_\pm$. We normalise

$$\psi_n^{0\,\dagger}\psi_n^0 = 1 \qquad \Longrightarrow \qquad \alpha_\pm^2 + \beta_\pm^2 = 1\,. \tag{3.10}$$

The 4-dimensional solutions of the Dirac equation are of the form[2]

$$\psi_D = \psi_n^0 \mathrm{e}^{\pm iE(t-nz)}\,, \tag{3.11}$$

where we have set $E = k = k_z$, with $k > 0$, for a string aligned along the z-axis. The zero-modes are massless excitations which travel at the speed of light along the string. Fermions trapped on a vortex (anti-vortex) travel in the $+z$ $(-z)$ direction. Note that both particles and anti-particles move in the same direction along the string, the direction

[2]For a Dirac particle, let's call it η, getting its mass by coupling to ϕ instead of ϕ^*, the zero mode solutions are $\eta^0_{n=1} = \psi^0_{n=-1}$ and $\eta^0_{n=-1} = \psi^0_{n=1}$. It follows that for a vortex a fermion zero mode due to Yukawa coupling to ϕ moves in opposite direction as a fermion zero mode due to Yukawa coupling to ϕ^*.

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being determined by whether the Yukawa coupling in eq. (3.5) is to a vortex or anti-vortex. The projection operator in eq. (3.7) singles out one helicity state (determined in terms of α_+ and β_+ or α_- and β_-). The zero mode solution has thus only two real degrees of freedom which correspond to particle and anti-particle. It is effectively a 2-dimensional Weyl fermion.

Now consider a Majorana fermion getting its mass by coupling to ϕ^*. The lagrangian reads

$$\mathcal{L}_M = \overline{\psi_L} i\gamma^\mu D_\mu \psi_L - \frac{1}{2}\lambda\phi\overline{\psi_R^c}\psi_L - \frac{1}{2}\lambda\phi^*\overline{\psi_L}\psi_R^c\,. \tag{3.12}$$

The equations of motions are

$$\begin{aligned} i\gamma^\mu D_\mu \psi_L &= \lambda\phi\psi_R^c\,, \\ i\gamma^\mu D_\mu \psi_R^c &= \lambda\phi^*\psi_L\,. \end{aligned} \tag{3.13}$$

Normalising the $\mathrm{U}(1)_X$ charge of ϕ to unity, the charges of ψ_L and ψ_R^c are $q_L = -q_R^c = -1/2$. The two equations of motion are not independent, rather the second equation in eq. (3.13) can be obtained from the first one by charge conjugation. The solution to the equation of motion is

$$\psi_M^{\mathrm{zm}} = \frac{1}{\sqrt{2}}(\psi_n^0(r,\theta)\mathrm{e}^{-iE(t-nz)} + \psi_n^{0*}(r,\theta)\mathrm{e}^{+iE(t-nz)}) \tag{3.14}$$

with ψ_n^0 as given in eq. (3.8) with the additional constraint that

$$\alpha_\pm(r,\theta) = \mp\beta_\pm(r,\theta)\,. \tag{3.15}$$

The Majorana solution is just a linear combination of the Dirac solutions given in eq. (3.11) which satisfies the reality constraint of eq. (2.8). There is only one solution corresponding to one real degree of freedom, which is half the number of degrees of freedom of the Dirac solution. The zero mode is effectively a 2 dimensional Majorana-Weyl fermion. A consequence of eq. (3.15) is that both the vector and axial currents are zero. Nevertheless there is energy-momentum flowing along the string.

3.2 Quantisation and conserved currents

In this section we quantise the Dirac and Majorana zero mode fields, and evaluate the conserved charges and energy-momentum.

As before, we start with the Dirac spinor. We use the classical wave solutions to the Dirac equation, eqs. (3.8), (3.11), to write the quantised field

$$\psi_n = \int_0^\infty \frac{\mathrm{d}k}{(2\pi)}\left(a(nk)\psi_n^0 e^{-ik(t-nz)} + b^\dagger(nk)\psi_n^0 e^{ik(t-nz)}\right), \tag{3.16}$$

with as before $k \equiv k_z$. Note that $k > 0$ always. Thus $a(k<0)$ are annihilation operators for states moving in the $-z$ direction, which can only be exited for an anti-vortex ($n=-1$). Likewise $a(k>0)$ are annihilation operators for states moving in the $+z$ direction, which can only be exited for a vortex ($n=1$). The wave function ψ^0 is k-independent by virtue of separation of variables in the equation of motion; the normalisation is also momentum independent.

Imposing the equal time anti-commutation relations

$$\{\psi(x,y,z),\psi^\dagger(x,y,z')\} = \delta(z-z')\psi_0\psi_0^\dagger\,, \tag{3.17}$$

it follows that the operators a, b satisfy the anti-commutation relations $\{a(k),a^\dagger(k')\} = \{b(k),b^\dagger(k')\} = 2\pi\delta(k-k')$. One can construct a Fock space in the usual way [17].

The vector current $j^\mu = \bar\psi\gamma^\mu\psi$ and axial current $j^{\mu 5} = \bar\psi\gamma^\mu\gamma^5\psi$ are identically zero for $\mu = 1,\, 2$. The non-zero components for $\mu = 0, 3$ are related, $j^3 = nj^0$ and $j^{35} = nj^{05}$, by virtue of eq. (3.7). The conserved charges corresponding to the vector and axial currents, $Q = \int \mathrm{d}^3x \langle :j^0:\rangle$ and $Q^5 = \int \mathrm{d}^3x\langle :j^{05}:\rangle$ respectively, are

$$\begin{aligned} Q &= \int_0^\infty \frac{\mathrm{d}k}{2\pi}\langle a^\dagger(k)a(k) - b^\dagger(k)b(k)\rangle \\ &= N - \bar N\,, \end{aligned} \tag{3.18}$$

$$\begin{aligned} Q^5 &= \int_0^\infty \frac{\mathrm{d}k}{2\pi}\langle(-\alpha^2+\beta^2)a^\dagger(k)a(k) + (\alpha^2-\beta^2)b^\dagger(k)b(k)\rangle \\ &= (\alpha^2-\beta^2)(-N+\bar N)\,. \end{aligned} \tag{3.19}$$

where, to avoid notational cluttering, we have dropped the subscript n to indicate whether it concerns a vortex or anti-vortex. Here : : denotes normal ordering, i.e., putting all annihilation operators to the left; this procedure automatically subtracts the zero point contributions.

The energy momentum tensor

$$T_{\mu\nu} = \bar\psi i\gamma_\mu\partial_\nu\psi - \eta_{\mu\nu}\bar\psi i\gamma^\sigma\partial_\sigma\psi \tag{3.20}$$

is likewise only non-zero for $\mu,\nu = 0,3$. The various components are related: $T_{00} = T_{33} = -nT_{03}$ and lead to the zero mode contribution to the energy per unit length of the string (respectively to the tension):

$$E_F = -T_F = \int_0^\infty \frac{\mathrm{d}k}{2\pi}k\langle a^\dagger(k)a(k) + b^\dagger(k)b(k)\rangle\,, \tag{3.21}$$

The quantisation of the Majorana field goes analogous. We write

$$\psi_n = \int_0^\infty \frac{\mathrm{d}k}{(2\pi)}\left(a(nk)\psi_n^0 e^{-ik(t-nz)} + a^\dagger(nk)\psi_n^0 e^{ik(t-nz)}\right), \tag{3.22}$$

supplemented by the anti-commutation relation $\{a(k),a^\dagger(k')\} = 2\pi\delta(k-k')$. The creation and annihilation operators for the particle and the antiparticle are now identical. Formally the Majorana zero mode can be obtained from the Dirac zero mode by setting $\beta_\pm \to \mp\alpha_\pm$ and $b(k) \to a(k)$.

The calculation of the conserved currents gives the following. As for any Majorana particle, the vector current vanishes (a Majorana mass term is not invariant under a vector U(1)). The axial current is also zero, by virtue of eq. (3.15): the amount of left-chiral and right-chiral fields exited for each zero mode is equal. We thus find

$$Q = 0\,, \qquad Q^5 = 0\,. \tag{3.23}$$

This can also be found from eq. (3.19), by substituting $b(k) \to a(k)$ and $\beta \to \alpha$, as appropriate for a Majorana spinor. However, the energy-momentum tensor is non-zero for $\mu, \nu = 0, 3$, as can be seen from eqs. (3.20), (3.21). This gives a non-zero contribution to the string energy per-unit-length and the string tension proportional to the number of Majorana particles trapped on the string:

$$E_F = -T_F = \int_{k>0} \frac{\mathrm{d}k}{2\pi} k \langle a(k)^\dagger a(k) \rangle \,. \tag{3.24}$$

Before we go to a discussion of the effective 2 dimensional theory, a word on notation/nomenclature. In the following we will refer to Q as particle number, the number of particles minus anti-particles (it is zero in the Majorana case). The term excitation number will be reserved for the total number of excitations, particles plus anti-particles. The excitation number does not correspond to a conserved current. It can be inferred from energy-momentum.

3.3 Dimensional reduction

The zero-mode solutions effectively live in 2 dimensions. In this section we discuss how to dimensionally reduce from 4 to 2 dimensions.

Define the matrices

$$\Gamma^{0\pm} = \frac{1}{2}(\pm\gamma^0 + \gamma^3), \qquad \Gamma^{1\pm} = \frac{1}{2}(\gamma^1 \pm i\gamma^2)\,. \tag{3.25}$$

They satisfy the algebra $\{\Gamma^{i+}, \Gamma^{j-}\} = \delta^{ij}$ and $\{\Gamma^{i\pm}, \Gamma^{j\pm}\} = 0$, and thus act as creation and annihilation operators. Define the ground state ζ as the state which is annihilated by $\Gamma^{i-}\zeta = 0$ for all i. We can then construct all states in the spinor representation by acting with creation operators on the vacuum. In 4D these states can be labelled by (s_0, s_1), with $s_i = \pm 1/2$, corresponding to the state

$$\zeta^{(s)} = (\Gamma^{0+})^{s_0+1/2}(\Gamma^{1+})^{s_1+1/2}\zeta\,. \tag{3.26}$$

The zero mode solutions $\psi^0_\pm$, see eq. (3.8), in this language are

$$\begin{aligned} \psi^0_+ &= \alpha_+\left(-\frac{1}{2}, -\frac{1}{2}\right) + \beta_+\left(-\frac{1}{2}, +\frac{1}{2}\right) \\ \psi^0_- &= \alpha_-\left(+\frac{1}{2}, +\frac{1}{2}\right) + \beta_-\left(+\frac{1}{2}, -\frac{1}{2}\right). \end{aligned} \tag{3.27}$$

Under dimensional reduction from 4 to 2 dimensions

$$\mathrm{SO}(3,1) \to \mathrm{SO}(1,1) \times \mathrm{SO}(2) \sim \mathrm{SO}(1,1) \times \mathrm{U}(1) \tag{3.28}$$

the spinor decomposes as $4 \to 2+2$. The 2's are spinor representation of $\mathrm{SO}(1,1)$, which are charged under the action of the global $\mathrm{U}(1)$. We can construct the 2D spinor representation analogous to the 4D construction. There is only one pair of creation/annihilation operators, given by

$$\tilde{\Gamma}^{0\pm} = \frac{1}{2}(\pm\tilde{\gamma}^0 + \tilde{\gamma}^1)\,. \tag{3.29}$$

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Here and in the following we denote all 2D quantities by a tilde. For the 2D gamma matrices we use $\tilde{\gamma}^0 = \sigma_1$ and $\tilde{\gamma}^1 = i\sigma_2$. The 2D spinors are labelled by (s_0). Explicitly, $(1/2) = (1\ 0)^T$ which corresponds to a right moving fermion, and $(-1/2) = (0\ 1)^T$ corresponding to a left moving fermion.

We dimensionally reduce $(s_0, s_1) \to (s_0)_{s_1}$, so that right/left moving in 4D corresponds to right/left moving in 2D. The s_1 labels the charge under the global U(1).[3] Applying this to the zero mode wave function we get

$$\begin{aligned} \psi^0_+ &\to \tilde{\psi}^0_+ = \alpha_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{-1/2} + \beta_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{+1/2} \\ \psi^0_- &\to \tilde{\psi}^0_- = \alpha_- \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{+1/2} + \beta_- \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{-1/2} \end{aligned} \tag{3.30}$$

Each zero mode has 2 degrees of freedom (particle/antiparticle), and can be described by a single 2D complex Weyl fermion.

The quantised Weyl spinor is

$$\tilde{\psi}_W = \int_0^\infty \frac{\mathrm{d}k}{2\pi} \left(\tilde{\psi}^0_n \tilde{a}(nk) \mathrm{e}^{-ik(z-nt)} + \tilde{\psi}^0_n \tilde{b}^\dagger(nk) \mathrm{e}^{ik(z-nt)} \right) \tag{3.31}$$

with $\tilde{\psi}^0_n$ given in eq. (3.30). We can now compute the 2D conserved charges. It is important to distinguish between 2D and 4D chirality. In D dimensions, chirality refers to left and right-handed particles, eigenstates of $P_{L,R} = (1_D \pm \gamma^5_D)/2$. The zero mode solution is a superposition of both 4D chiralities. In 2D $\tilde{\gamma}^5 = \mathrm{diag}(-1, 1)$. Hence, 2D chirality is equivalent to left and right moving particles, which is unrelated to the chirality concept in 4D. In 2D the vector and axial currents are related $\tilde{j}^{\mu 5} = \epsilon_{\mu\nu} \tilde{j}^\nu$. The two conserved charges are

$$\begin{aligned} \tilde{Q}_{n=1} &= N_{\tilde{L}} - \bar{N}_{\tilde{L}} \qquad \text{and} \qquad \tilde{Q}^5_{n=1} = -(N_{\tilde{L}} - \bar{N}_{\tilde{L}}) \\ \text{or} \quad \tilde{Q}_{n=-1} &= N_{\tilde{R}} - \bar{N}_{\tilde{R}} \qquad \text{and} \qquad \tilde{Q}^5_{n=-1} = N_{\tilde{R}} - \bar{N}_{\tilde{R}} \end{aligned} \tag{3.32}$$

with $\tilde{L}, \tilde{R}$ denoting 2D chiralities, i.e., left/right moving. In 4D left/right moving is related to fermion getting their mass via coupling to ϕ or ϕ^*. 2D particle number $\tilde{Q}$ corresponds to 4D particle number Q. The 4D axial current corresponds to the 2D global U(1) current, under which states with different s_1 are charged differently. Furthermore, the 2D energy and momentum are equivalent to the 4D expressions.

The 2D Majorana fermion is obtained by setting $\alpha = \beta$ in eq. (3.30). It has 1 real d.o.f., corresponding to a Majorana-Weyl fermion. The 2D charge conjugation matrix is $C = \mathbf{1}_2$. In addition, under charge conjugation also $s_1 \leftrightarrow -s_1$. There is no global U(1) symmetry, no vector or axial charges but there is a non-zero energy and momentum.

[3]This can formally be implemented by introducing a matrix $U = (1_2\ \sigma_1)^T$ such that $U^\dagger \Gamma^{0+} U = \tilde{\Gamma}^{0+}$. Then $\tilde{\psi} = U\psi$ is the same operation as $(s_0, s_1) \to (s_0)$. Helicity eigenstates, i.e., eigenstates of $\gamma^0\gamma^3\gamma^5\psi = \pm\psi$, have $s_1 = \pm 1/2$.

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3.4 Bosonisation

If is well know that a 2D fermionic theory with Dirac fermions is equivalent to a 2D bosonic theory. Here, we remind the reader how fermionic superconductivity can be discussed in terms of bosonic superconductivity [11].

If there are both left and right moving fermionic zero modes the theory is anomaly free if the sum of charges of the left moving particles is equal to those of the right moving particles. A theory with two zero modes of 4D Dirac particles moving in opposite directions is effectively equivalent to two 2D Weyl fermions with opposite chirality with the same charge. These combine to form a single 2D Dirac fermion. Such a theory can be bosonised. If the fermion is charged with charge e, it is equivalent to a massive scalar boson with mass $e/\sqrt{\pi}$.

If on the other hand there is only a left (or right) moving Dirac zero mode, which is necessarily neutral, the effective theory consists of a single 2D Weyl fermion which has 2 real degrees of freedom. It is equivalent to a free massless chiral boson with an action

$$\mathcal{L}_B = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\,. \tag{3.33}$$

subject to the constraint:

$$(\partial_t \pm \partial_z)\phi = 0\,. \tag{3.34}$$

The bosonic action in eq. (3.33) is invariant under $\phi \to \phi + n$, with n integer (the integer comes from the integer valued fermion current). Hence, ϕ is an angular variable. A topological invariant can be constructed for a string loop $Q = \int \mathrm{d}l(\partial_z\phi)$ [11]. The topological invariant Q corresponds to particle number $Q = N - \bar{N}$ of the fermionic system.

A Majorana zero mode has only one real d.o.f. and can be described by a 2D Majorana-Weyl fermion. Bosonisation is not possible, as a Majorana-Weyl spinor has central charge $c = 1/2$, whereas a chiral boson has central charge $c = 1$ — the degrees of freedom do not match. Vortons studies are commonly done in terms of the bosonic description and it is not clear that this is adequate for a Majorana.

3.5 Discussion

The difference between the neutral Dirac current and the Majorana current is the following.

For the Dirac case particle number is non zero and flowing along the string. It is a conserved quantum number. Current conservation is guaranteed by an (accidental) U(1) quantum number (such as lepton number). Two leptons cannot pair annihilate, and one lepton cannot decay into a state without lepton number. A left (right) moving zero mode can be described by a left (right) moving Weyl fermion. The action can be bosonised, and is equivalent to a chiral boson. One can define a winding number which is conserved. This topological invariant can be defined for the boson field which corresponds to particle number in the fermionic description.

If there is only one fermionic zero mode the string is chiral. This can only happen for neutral fermions, otherwise there would be an anomaly. Although the current might be persistent, it does not correspond to superconductivity, in the sense that there is no

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Meisner effect. Moreover, in the fermionic language, there are no cooper pairs. Cooper pairs are bound states of fermions with opposite spin and momenta, which therefore can only form if there are both left and right moving currents [18].

For the Majorana case, there is no chiral or vector charge. Thus no-topological invariant can be defined. This is related to the fact that a 2D theory with an odd number of Majorana-Weyl fermions cannot be bosonised. Particle number is zero but in principle there can be a non-vanishing number of particles: there is then an energy-momentum flow. Since the Majorana does not carry a conserved quantum number, the number of Majorana excitations is not a conserved quantity in general. However, massless excitations cannot decay, neither pair annihilate (as they do not scatter), and thus the current is stable for kinematical reasons in the massless limit.

Note that the masslessness of the zero mode is only a good approximation for an infinitely long straight string. String curvature and string oscillations will induce an effective mass. This will be discussed in more detail in the next section in the context of vorton formation. The other issues we want to address is whether a non negligible amount of chiral fermions can actually be trapped as transverse zero modes on the string, and contribute to the string energy-tension, and what is the actual difference between a flow of Majorana-Weyl fermions and a current of neutral Weyl fermions.

4. Vortons

In this section we discuss fermionic superconductivity in cosmic strings. The first ingredient we need is a fermionic zero mode. In the previous section, we have discussed the zero mode in detail for both Dirac and Majorana particles. Once we have a model which admits zero mode solutions, we need to know whether these zero modes can be exited or can be trapped so as to generate a current. Then finally, we must check for the classical and quantum stabilities of these currents.

In the case of charged particles, a current can be generated as the string moves through a magnetic field [11]. This is not possible for neutral particles. The only way that a large loop current can be generated is if there is an initial tiny current. As the loop collapses, the line integral of the current remains constant so that the current grows as the inverse of the loop length. In the case of charged particles, there are both left and right moving modes in order to cancel anomaly; charged fermions can thus scatter and this process strongly limits the current [13]. In the case of a neutral particle, there is only left or right moving modes, and the scattering process is absent [13]. Therefore large currents may only occur if there is an asymmetry between left and right moving modes. This chiral asymmetry can only happen for neutral fermions which can either be Dirac or Majorana.

Neutral currents cannot induce any (large scale) electric or magnetic field. Their cosmological interest lies solely in the conjecture that fermionic currents can stabilise string loops against gravitational collapse. Such stable, superconducting string loops are called vortons [12]. Vortons contribute to the dark matter in the universe. The requirement that they do not overclose the universe can put strong constraints on the underlying particle physics [19].

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Vortons can be understood semi-classically [12, 14]. Consider a fermionic current flowing along a string loop. Assume that the loop curvature does not significantly alter the zero mode spectrum obtained in the limit of zero curvature. To simplify all formulas, we further assume that for the Dirac current $N \gg \bar{N}$.[4] The zero mode energy levels are $\epsilon_n = n/R$, with R the radius of the string loop. Each level can be occupied by at most one fermion and one anti-fermion, in accordance with Pauli exclusion principle. In the ground state, the highest occupied level has Fermi energy $\epsilon = N/R$. The total fermionic energy is obtained by summing over all occupied levels. The total energy of string is the sum of energy in bosonic and fermionic excitations $E = E_{\rm B} + E_{\rm F}$:

$$E = 2\pi\mu R + \frac{N^2}{2R} \tag{4.1}$$

where we have set $N(N+1) \to N^2$, valid in the limit $N \gg 1$. Further $\mu \sim \phi_0^2$ is the string energy per unit length. As the loops shrinks under gravitational action the bosonic energy decreases, but the fermionic energy increases due to an increasing spacing between energy levels. The vorton radius, the radius for which the energy is minimised, is

$$R_0 = \frac{N}{\sqrt{4\pi\mu}}\,. \tag{4.2}$$

The above semi-classical description is valid for a loop radius larger than the string width and than the Compton wavelength of the zero mode. In particular, the last requirement, $R_0 > m_\psi^{-1}$, gives

$$N > \lambda^{-1}\,, \tag{4.3}$$

with λ the Yukawa coupling, see eqs. (3.5), (3.12). We will discuss current formation and stability for Dirac and Majorana currents, which may result in vorton formation, in the rest of this section.

4.1 Effective zero mode mass

To understand vorton formation, to understand current formation and stability, it is important to realise that the strings in the early universe are not infinitely long straight strings. Rather, the strings vibrate and curve, there are cusps and intersections. The zero-mode, which is a strictly massless excitation (in the sense that it travels at the speed of light) only for an infinitely long straight string, obtains an effective mass under such conditions. Processes which are absent for massless zero modes, decay is an obvious example, become possible. It goes without saying therefore, that an effective mass for the zero mode can alter the formation and stability of the fermionic currents considerably. Before considering these processes in more detail, we will first discuss under which conditions the zero mode gets effectively massive.

[4] For $N \sim \bar{N}$ we find that the Fermi level is $\epsilon = \max[N, \bar{N}]/R$ and fermionic energy $E_f = (N^2+\bar{N}^2)/(2R)$, results which only differ by order unity from those for $N \gg \bar{N}$. In the literature vortons are often discussed in bosonised language, in which the only quantum number is particle number $Q = N - \bar{N}$, which can be substantially different from $N + \bar{N}$.

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The zero-mode lives in a potential well of height $E = m_\psi$. At the bottom of the well the excitation is massless. What keeps the zero mode in a circular orbit for a string loop is that as the mode strays off the centre of the vortex (bottom of the well) it becomes massive. There is a restoring force (the centripetal force) $-\partial V/\partial r \sim -\delta m/\delta r$ to keep it in orbit. The effective mass of the zero mode excitation can be estimated as follows [20]. The acceleration of a particle going at the speed of light around a circle with radius R is $|\dot{k}| = k/R$. Setting this equal to the centripetal force, and reminding that $\delta r \sim m_\psi^{-1}$ with m_ψ the vacuum fermion mass (since the zero mode wave function falls off as $e^{-m_\psi r}$ at large r) we get

$$m \sim \frac{\dot{k}}{m_\psi} \sim \frac{k}{R m_\psi}\,. \tag{4.4}$$

The higher the momentum of the mode the larger the effective mass.

Another way to see that string curvature induces an effective mass (alters the dispersion relation to $E \neq |\vec{k}|$) is to consider the solutions to the fermionic equations of motion. Consider a string loop with radius R lying in the xy-plane, with its centre at the origin. Define also spherical coordinates (r', θ') which parameterise the plane transverse to the string, with the origin at the string core. This plane is characterised by $\vec{r'} \times \vec{\theta'} \parallel \vec{k}$, and $A_\perp = 0$. The zero mode solution is schematically of the form

$$\psi = \mathrm{e}^{\pm i(Et + \vec{k}\cdot\vec{x})} \psi_0(r', \theta')\,. \tag{4.5}$$

Note that both $\vec{k}$ and $\vec{r'} \times \vec{\theta'}$ are functions of time, since as the zero mode travels along the string loop $\vec{k}/|\vec{k}|$ changes. As a result the ∂_t operator in the equation of motion not only brings down a factor E, but also terms proportional to $\dot{k}/k$. This gives a dispersion relation of the form

$$E \sim k + \frac{\dot{k}}{k}\,. \tag{4.6}$$

Expanding $E = \sqrt{k^2 + m^2} \approx k + m^2/k$ valid for $m \ll k$, and setting $\dot{k} \sim m m_\psi$ equal to the centripetal force, this gives the same mass correction as the classical argument leading to eq. (4.4).

Another string configuration which leads to an effective mass for the zero modes is a vibrating string. Consider a string extended along the z-direction, which moves with some velocity in the x-direction. The dispersion relation for the zero mode is $E^2 = k_x^2 + k_z^2$. By performing a Lorentz transformation we can go to a frame in which the string is at rest, and $E = k_z$. However, for a vibrating string we cannot go to a frame in which the string is at rest at all times, as $\dot{k} \neq 0$. Just as for the case of a string loop, the dispersion relation picks up a term $\dot{k}/k$, see eq. (4.6). For string loops the natural vibration frequency is $\omega \sim 1/R$, whereas for string vibrations the natural scale is $\omega \sim \sqrt{\mu} \sim \phi_0$. The former case gives an effective mass as in eq. (4.4), whereas the latter gives a mass which is larger by a factor $\phi_0 R$.

Finally, effective mass terms are generated if locally components of the gauge field other than A_θ are excited, or if the Higgs field profile deviates from its cylindric symmetric form. The string profile can be disturbed in string intersections or in cusps. A non-zero

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A_z or A_0 component leads to a dispersion relation of the form

$$E = k + qA \,, \tag{4.7}$$

and the effective mass is $m^2 = qAk$. A non-zero A_r (or non-cylindrical component of ϕ) will appear as an effective mass term in the r, θ dependent part of the equation of motion. It can be removed from the equation of motion by multiplying the zero mode wave function by a factor e^{iqAz}. It thus likewise gives a dispersion relation of the form eq. (4.7). In cusps and intersections $A \sim 1/r \sim m_f$ in eq. (4.7), and the effective mass is large $m \sim k$.

The fermionic currents back react on the gauge field. This creates an effective mass for non-chiral strings but not for chiral strings. [17]

4.2 Current formation

We consider chiral strings only (as appropriate for Majorana currents) and thus chargeless current carriers only. There is no possibility of current creation through spectral flow, as in the original Witten model, as the fermionic zero modes are neutral. Hence, any net current must be the result of fermions getting trapped/condensing on the string. Currents may be created in thermal or non-thermal settings.

Thermal trapping. Let's first consider the case in which the string is emerged in a thermal bath with a reheat temperature larger than the vacuum fermion mass $T_R > m_\psi$. Whatever the specifics of the trapping mechanism, it is expected to be inefficient at high temperature $T \gg m_\psi$. The reason is that the typical energies involved in the trapping reactions are much higher than the binding energy of the zero mode $E_b = -m_\psi$, and trapping is suppressed. Fermions that do get captured by the string have typical momenta $k \gg m_\psi$ and can escape easily at cusps and intersections. Furthermore, any zero mode present with $k \lesssim m_\psi$ can be easily scattered off the string. Hence, at high temperature we expect the number of zero modes to be negligible small. Current condensation has to take place at temperatures $T \lesssim m_\psi$.

On the other hand, at temperatures $T \ll m_\psi$ capturing processes are also suppressed. For processes with one or more free fermions ψ in the in-state this is obvious, as their number density is Boltzmann suppressed. But also if the initial state contains only light particles there is a suppressions factor. The reason is that the typical wavelength probed in the reaction is much larger than the width of the zero mode, and the overlap between the zero mode wave function and those of the initial particles is small. As a result reaction rates are smaller by a factor $(E/m_\psi)^2$ compared to the rate expected on dimensional grounds, with $E \sim T$ the typical interaction energy (see appendix [21, 22, 23]).

We thus conclude that current formation, interactions in which zero modes are captured/produced, has to take place at $m_\psi \sim T$.

Let us now consider the possible processes in which zero modes can be produced. The first interaction that comes to mind is a fermion scattering off the string to produce a zero mode final state, that is, the reaction $\psi \to \psi_{\rm zm}$ in the string background. The string is just a static background, it cannot exchange longitudinal momentum or spin with the fermionic states. Therefore energy and longitudinal momentum cannot be conserved in the reaction if the mass of the incoming fermion is larger than the effective mass m of the zero mode, see eq. (D.9). And thus zero mode production in fermion-string scattering does not occur.

But there are other reactions possible through which zero modes can be produced. If the only interactions present in the theory are those given in eqs. (3.5), (3.12), these are of the form $\psi + \psi \to \psi_{\rm zm} + \psi$.[5]

In the limit $T \sim m_\psi$ the capturing and scattering rate are of the same order of magnitude, and if the reaction is sufficiently fast, a thermal equilibrium distribution of zero modes will be established. We thus define the number of zero modes on a loop of length L by

$$N \sim N + \bar{N} = p \frac{L}{\xi} \,. \tag{4.8}$$

with $\xi \sim T^{-1}$ the thermal correlation length. The efficiency factor p is of order unity if thermal equilibrium is reached, and $p \simeq 0$ if trapping reactions are frozen out at $T \sim m_f$. The average particle number over many loops is zero $\langle N - \bar{N} \rangle = 0$. However, the variance, which gives the order of magnitude of particle number on the individual loop is non-zero: $\sqrt{\langle (N - \bar{N})^2 \rangle} \sim \sqrt{N}$.

Thermal equilibrium can be established if the reaction rate is sufficiently fast compared to the Hubble rate $\Gamma \sim n_\psi \sigma > H$. The thermal fermion density is $n_\psi \sim T^3 \sim m_\psi^3$ at the temperature of interest. The cross section for the four-fermion interaction at energies $T \sim m_\psi$ is derived in appendix D.2. It is

$$\sigma \sim g_X^4 A^6 \frac{m_\psi^2}{\max[m_X, m_\psi]^4} \,. \tag{4.9}$$

As discussed in the appendix we expect the amplification factor A to be of order unity. If the reaction is mediated by the Higgs field $m_X = m_\phi$ and $g_X = \lambda$ is the Yukawa, whereas if the reaction is gauge mediated we have $m_X = m_A$ and $g_X = g$ the fermion gauge coupling. The reaction rate exceeds the Hubble rate for

$$\frac{m_X}{g_X A^{3/2}} < (m_\psi^3 M_{\rm pl})^{1/4} = \begin{cases} 3 \times 10^{16}\,{\rm GeV} \;\; \text{for } m_\psi \sim 10^{16}\,{\rm GeV} \\ 1 \times 10^{15}\,{\rm GeV} \;\; \text{for } m_\psi \sim 10^{14}\,{\rm GeV} \\ 3 \times 10^{13}\,{\rm GeV} \;\; \text{for } m_\psi \sim 10^{12}\,{\rm GeV} \end{cases} \tag{4.10}$$

If $m_X < m_\psi$, one should replace $m_X \to m_\psi$ in the above equation. If indeed $A \sim 1$ and $g_X \lesssim 1$ this becomes impossible as $m_\psi \to 10^{16}$ GeV.

It is possible that the zero mode fermion ψ couples to lighter fermions χ, for example, through an interaction $\mathcal{L}_I = hH\bar{\psi}_R \chi_L + {\rm h.c.}$ This is of the form of the coupling of the right-handed neutrino to the left-handed one. Such a coupling allows for trapping reactions $\chi + \chi \to \psi_{\rm zm} + \psi$ mediated by a light Higgs field.[6] At energies $T \sim m_\psi$ (threshold energy) the cross section is $\sigma \sim h^4 m_f^{-2}$. This is of course of the same form as the cross section eq. (4.9) in the limit $m_X < m_f$, and thus the bound on h derived below applies also to g_X in

[5] An interaction such as $\psi + \phi \to \psi_{\rm zm}$, possible at $T \sim m_\psi$ if $m_\phi < m_\psi$ has infinitely small phase space as the spin of the incoming fermion has to match that of the zero mode.

[6] The reaction $f' + h \to f_{\rm zm}$ is phase space suppressed, as the incoming light fermion needs to have the same spin as the zero mode.

this limit. The trapping rate is $\Gamma \sim h^4 m_\psi$, which exceeds the Hubble rate at temperatures $T \sim m_\psi$ for

$$h > \left(\frac{m_\psi}{M_{\rm pl}}\right)^{1/4} = \begin{cases} 0.3 \ \text{ for } m_\psi \sim 10^{16}\,\text{GeV} \\ 0.1 \ \text{ for } m_\psi \sim 10^{14}\,\text{GeV} \\ 0.03 \ \text{ for } m_\psi \sim 10^{12}\,\text{GeV} \end{cases} \tag{4.11}$$

Large couplings are needed for effective current formation.

We conclude that thermal current condensation can only takes place at $T \sim m_\psi$, and only if the trapping reactions are fast enough. This last requirement is hard to satisfy in the limit $m_\psi < m_A, m_\phi$ and in the absence of large Yukawa couplings to light fermions. If large Yukawa are present in the theory, current formation is expected to happen for m_ψ sufficiently small, with the number of zero modes given by eq. (4.8).

For Dirac spinors lepton number is substantially less than the the total number of zero modes: $Q = N - \bar{N} \sim \sqrt{N + \bar{N}}$. Since massless particles do not annihilate, the current description should be in terms of excitation rather than particle number. In the literature, vorton formation is often discussed in bosonised language in which the only quantum number is lepton number. The bosonic language strictly only applies if all particle anti-particle pairs have already annihilated, which would mean zero current for the Majorana case.

Non-thermal capturing Fermionic current condensation can also happen non-thermally. If there is a large amount of energy released inside the string, this will create bosonic excitations (A and ϕ), which decay into fermions. If decay is immediate, $\Gamma_{\rm dec} > H$ evaluated at the time of the energy release, the bosons decay inside the string, and the decay fermions may be captured as zero-modes. This was first discussed in [24], where they assume the energy-release is due to an internal (only inside the string) phase transition. This needs a complicated Higgs sector, and we will not discuss it further.

One could also consider the case $T_R < m_\psi$. Before decay the Higgs field oscillates in its potential. If $\dot{w} < w^2$ with $w = k^2 + m_{\rm eff}^2(\phi)$ for all particles coupling to the Higgs field (ϕ, A, ψ), then the oscillation is adiabatic. The string profile function and the fermion zero modes change adiabatically to adapt to the changing background fields. However, in the opposite limit, the evolution is non-adiabatically. Parametric resonance leads to explosive particle production. The fermions thus produced inside the string (from direct amplification, or from decay of the bosonic fields) can be trapped. It is not clear whether this mechanism can be efficient. One suppression factor is that only soft quanta are exited during preheating with typical wave numbers k_* much larger than the string width. Trapping is then suppressed by factors (k_*/m_ψ).

To summarise, chiral currents which are necessarily neutral cannot be created by spectral flow. Current formation can still take place, either thermally or non-thermally. However, both processes are not obvious, and require special circumstances.

4.3 Current stability

The effect of an (momentum dependent) effective mass opens up the possibility for zero modes leaving the string, for pair annihilation, and for zero mode decay. We will discuss these processes in turn.

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Classically, the zero mode cannot leave the string as long as $m < m_\psi$. Conservation of angular momentum requires the massive mode outside to take away all the momentum of the zero mode. Consequently, there is an energy barrier if $m < m_\psi$, but not in the opposite limit. As first noted in [20], the generation of an effective mass as in eq. (4.4) puts a maximum on the current. For a vorton loop $m \sim k/(R_0 m_\psi)$ with R_0 given in eq. (4.2). Using that the zero mode momentum is typically of the order of the Fermi momentum $k_f \sim N/R_0$, the typical mass is

$$m \sim \frac{4\pi}{N}\frac{\phi_0^2}{m_\psi} \sim 10^{14}\,\mathrm{GeV}\left(\frac{10^5}{N}\right)\left(\frac{\phi_0}{10^{16}\,\mathrm{GeV}}\right)^2\left(\frac{10^{14}\,\mathrm{GeV}}{m_\psi}\right). \tag{4.12}$$

Requiring $m < m_\psi$ so that the current is classically stable gives

$$\lambda^2 N > 4\pi\,, \tag{4.13}$$

where we have used that $m_\psi = \lambda\phi_0$. For perturbative couplings this gives a stronger constraint than eq. (4.3). Although classically the zero mode cannot leave the string if $m < m_\psi$, there is a non-zero tunnelling probability for this process to happen [14]. The extra energy needed to overcome the energy barrier can be provided by the string: enlarging the loop radius will increase the bosonic energy, see eq. (4.1). The current is stable against tunnelling processes (lifetime longer than age of the universe) for [14]

$$N \gtrsim \frac{100}{\alpha^{3/2}} \tag{4.14}$$

with $\alpha = g^2/(4\pi)$ the fine-structure constant of the broken $\mathrm{U}(1)_X$.

Another consequence of a non-zero mass is the possibility of zero mode decay. As we have seen, thermal current formation is most effective if the zero mode has a large, order unity, coupling to light particles. Decay is kinematically allowed if the zero mode is heavier than the decay products. The decay rate for a coupling h, see Eq (C.13), then is

$$\Gamma \sim \frac{h^2 m^3}{m_\psi^2}\left(1 + \mathcal{O}\left(\frac{m}{m_\psi}\right)^2\right). \tag{4.15}$$

For $m \ll m_\psi$ the Compton wavelength m^{-1} is much smaller than the width of the zero mode, and the wave function overlap of the zero mode and the decay products is small. This gives a suppression factor $(m/m_\psi)^2$. For order one couplings and masses $m \gtrsim 10\,\mathrm{GeV}$, decay occurs before big bang nucleosynthesis (BBN). It follows that current stability can only be assured if the decay is kinematically forbidden and $m < m_{\mathrm{dec}}$.

A lower bound on the decay width comes from gravitational decay

$$\Gamma \sim \frac{m^3}{M_{\mathrm{pl}}^2}\left(\frac{m}{m_\psi}\right)^2 \tag{4.16}$$

where we have again included a suppression factor $(m/m_\psi)^2$. Decay happens before BBN, and thereby evading all bounds from vorton overdensities, if

$$m \gtrsim 10^8\,\mathrm{GeV}\left(\frac{m_\psi}{10^{14}\,\mathrm{GeV}}\right)^{2/5}. \tag{4.17}$$

This is generically satisfied for high scale phase transitions unless N is extremely large. Decay of Majorana particles cannot be suppressed by invoking symmetries; decay of Dirac particles can be suppressed by lepton number conservation, although it is believed that gravity violates all global symmetries.

A third effect of a non-zero mass is that different modes travel at different velocities and can scatter off each other. Particles and anti-particle states (Dirac case) or two particles can pair annihilate (Majorana case) into decay products whose masses are lighter than the Fermi energy. The typical centre of mass energy is of the order of the zero mode mass. Thus annihilation is only possible if the mass of one of the zero modes is larger than the masses of the decay products. If eq. (4.13) is violated, not only can zero modes leave the string, also the reaction $\psi_{\rm zm} + \psi_{\rm zm} \to \psi + \psi$ becomes possible. Scattering into lighter particles, $\psi_{\rm zm} + \psi_{\rm zm} \to X$ is kinematically accessible for $m_X < m_\psi$. The conditions under which scattering is effective are similar to those under which zero mode decay occurs. Note that pair annihilation into light particles cannot be forbidden by lepton number. If it does happen, the Majorana current is destroyed, whereas a Dirac current can subsist if there is a net lepton number.

To summarise, the zero-modes obtain an effective mass from string loop curvature, string vibrations, and string perturbations making escape, pair annihilation and decay of the zero modes possible. Escape can be prohibited for large enough N. Annihilation and decay processes might be suppressed or forbidden for Dirac fermions by conservation of lepton number (though gravity is believed to violate global quantum numbers), but not for Majorana fermions. Decay and annihilation can occur at relatively low Fermi levels if the zero modes couple to light fermions — which is precisely the favoured condition for thermal current formation. But even in the absence of light fermions, gravitational decay happens before nucleosynthesis unless N is very large or the string forming phase transition happens at low scale. Vorton formation does not look probable, especially for Majorana fermions.

5. Conclusions

In this paper we have derived the zero mode solution for a Majorana neutrino and compared it with its Dirac counterpart. The Majorana zero mode has one degree of freedom, one less than the Dirac zero mode. Both the vector and axial currents are zero. It is therefore impossible to bosonise the 2D effective action for a Majorana zero mode.

No conserved quantum number can be defined to assure the stability of a Majorana zero mode. This is in contrast with Dirac fermions, for which always the analogue of lepton number can be introduced. However, massless zero modes are stable for kinematical reasons. And in the massless limits there is thus no difference between the stability of Majorana and Dirac zero modes.

The zero mode is only strictly massless for infinitely long straight strings. The cosmic strings in our universe are expected to be curved, to vibrate, to fold and to intersect. Under such conditions an effective mass for the zero mode is generated. Needless to say that an effective mass alters the stability properties of a zero mode. We have studied these issues in the context of vortons, string loops stabilised by a fermionic current.

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Neutral fermionic currents can only be formed if zero modes are somehow trapped on the string. A massive fermion outside cannot simply be trapped through scattering with the string, as this would violate energy-momentum conservation. Interactions in which more than two initial/final state particles participate are needed. Such interactions can only be efficient if the zero mode couples to light particles with large, order one, couplings.

The fermionic current on a string loop increases as the loop contracts. The effective zero mode mass, which is proportional to the loop curvature, increases in this process, opening up the possibility of zero mode decay. The decay width is large if the zero mode has a large coupling to light particles, precisely the conditions under which current formation is effective. We expect that vortons will not form unless the string scale is low. This is especially true for Majorana neutrinos which can pair annihilate, and whose decay cannot be protected by a conserved quantum number.

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A. Notation

We use the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. For the gamma matrices we take chiral basis:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.1}$$

with $\sigma^\mu = (1, \sigma^i)$ and $\bar\sigma^\mu = (1, -\sigma^i)$, and σ^i the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1. \end{pmatrix} \tag{A.2}$$

In this basis, the projection operators $P_{\mathrm{L,R}}$ are defined by

$$\psi = \psi_L + \psi_R = P_{\mathrm{L}}\psi + P_{\mathrm{R}}\Psi = \frac{1-\gamma^5}{2}\psi + \frac{1+\gamma^5}{2}\psi\,. \tag{A.3}$$

The charge conjugate of a spinor is defined as

$$\psi^c = C\psi^\dagger = \tilde{C}\bar\psi^T \qquad \text{and} \qquad \overline{\psi^c} = \psi^T\tilde{C}\,, \tag{A.4}$$

with

$$\begin{aligned} C &= i\gamma^2 1,, \\ \tilde{C} &= C(\gamma^0)^T = i\gamma^2\gamma^0\,. \end{aligned} \tag{A.5}$$

The latter matrix has the properties $\tilde{C} = -\tilde{C}^{-1} = -\tilde{C}^\dagger = -\tilde{C}^T$. The charge conjugate field ψ^c acts similarly as the hermitian conjugate field $\bar{\psi} = \psi^\dagger\gamma^0$: it creates particles and annihilates anti-particles. The chiral projections of ψ^c are given by

$$\psi^c_{L,R} = \tilde{C}\bar{\psi}^T_{R,L} \qquad \text{and} \qquad \bar{\psi}^c_{L,R} = \psi^T_{R,L}\tilde{C}\,. \tag{A.6}$$

Note that $\psi^c_L \equiv (\psi^c)_L = P_L\psi^c = P_L\tilde{C}\bar{\psi}^T = \tilde{C}P_L\bar{\psi}^T = \tilde{C}(\bar{\psi}P_L)^T = \tilde{C}\bar{\psi}^T_R = (\psi_R)^c$ is a left-handed spinor, not to be confused with $(\psi_L)^c = (P_L\psi)^c = \tilde{C}\bar{\psi}^T_L = (\psi^c)_R \equiv \psi^c_R$ which is a right-handed spinor.

B. Spinor conventions

In this appendix we list our spinor conventions. The normalization is slightly different from that used in the paper.

B.1 Free Dirac spinor

We first give the results for a free massive Dirac spinor. The positive and negative frequency solutions of the Dirac equation are

$$u(p,s) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix}, \qquad v(p,s) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ -\sqrt{d\cdot\bar{\sigma}}\xi_s \end{pmatrix}, \tag{B.1}$$

with $r = \pm$ corresponding the two independent helicity/spin states. In our conventions $\xi_+ = \binom{1}{0}$ and $\xi_- = \binom{0}{1}$. The spinors are normalized to

$$u^\dagger(p,s)u(p,r) = v^\dagger(p,s)v(p,r) = 2E_p\delta_{rs}\,, \tag{B.2}$$

$$\bar{u}(p,s)u(p,r) = -\bar{v}(p,s)v(p,r) = 2m\delta_{rs}\,, \tag{B.3}$$

$$u^\dagger(p,s)v(-p,r) = v^\dagger(-p,s)u(p,r) = 0\,, \tag{B.4}$$

$$\bar{u}(p,s)v(p,r) = \bar{v}(p,r)u(p,r) = 0\,. \tag{B.5}$$

The quantized field is

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\Big(a_s(p)u(p,s)\mathrm{e}^{-ip\cdot x} + b^\dagger_s(p)v(p,s)\mathrm{e}^{ip\cdot x}\Big)\,. \tag{B.6}$$

Imposing an equal-time anti-commutation relation for ψ gives the anti-commutation relations for the creation and annihilation operators:

$$\{\psi,\psi^\dagger\} = \delta^3(\vec{x}-\vec{y})1_4 \quad \Longrightarrow \quad \{a_s(p),a^\dagger_r(k)\} = \{b_s(p),b^\dagger_r(k)\} = (2\pi)^3\delta^3(\vec{p}-\vec{k})\delta_{rs} \tag{B.7}$$

with 1_4 the unity matrix. We define the one-particle state as

$$|ps\rangle = \sqrt{2E_p}a^\dagger_s(p)|0\rangle\,. \tag{B.8}$$

so that

$$\langle ps|kr\rangle = 2E_p(2\pi)^3\delta^3(\vec{p}-\vec{k})\delta_{rs} \tag{B.9}$$

where we have used that the vacuum is normalized to unity $\langle 0|0\rangle = 1$. Our renormalization choice eq. (B.8) corresponds to a wave function normalization of $2E_p$ particles per unit volume. The wave function is

$$\langle 0|\psi|ps\rangle = u(p,s)\mathrm{e}^{-ip\cdot x}\,. \tag{B.10}$$

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B.2 Zero mode spinors

The classical zero mode solution to the wave equation is (we consider only $n = 1$, and coupling to ϕ^*)

$$\psi^0(p) = \begin{pmatrix} \alpha\sqrt{p\cdot\sigma}\xi_- \\ \beta\sqrt{p\cdot\bar{\sigma}}\xi^+ \end{pmatrix}. \tag{B.11}$$

This is the solution for both positive and negative frequencies. Further we require

$$\int \mathrm{d}x\,\mathrm{d}y\;(\alpha^2+\beta^2) = 1 \tag{B.12}$$

so that the spinors are normalized

$$\begin{aligned} &\int \mathrm{d}x\,\mathrm{d}y\;\psi^{0\dagger}(p)\psi^0(p) = 2E_p \\ &\bar{\psi}^0(p)\psi^0(p) = \psi^{0\dagger}(p)\psi^0(-p) = 0\,. \end{aligned} \tag{B.13}$$

We approximate the profile functions by

$$\alpha = -\beta = \frac{M}{\sqrt{\pi}}\mathrm{e}^{-Mr}\,, \tag{B.14}$$

satisfying the normalization condition eq. (B.12). The zero modes for a vortex/anti-vortex with $|n| = 1$ are independent of the angular variable θ. The width of the wave function is set by the fermion mass outside the string $M = m_\psi$,[7] as the solution to the Dirac equation at large r is $\propto \mathrm{e}^{-m_\psi r}$.

The quantized zero mode field is

$$\psi_{\mathrm{zm}} = \int \frac{\mathrm{d}p}{2\pi}\frac{1}{\sqrt{2E_p}}\left(a_p\psi_p^0\mathrm{e}^{-ip(t-z)} + b_p^\dagger\psi_p^0\mathrm{e}^{ip(t-z)}\right) \tag{B.15}$$

with $p \equiv |\vec{p}_L| = E_p$ with $\vec{p}_L$ the momentum longitudinal to the string. The field anti-commutator reads

$$\int \mathrm{d}x\,\mathrm{d}y\;\{\psi,\psi^\dagger\} \quad\Longrightarrow\quad \{a(p),a^\dagger(k)\} = \{b(p),b^\dagger(k)\} = (2\pi)\delta_L^1(\vec{p}-\vec{k}) \tag{B.16}$$

with $\bar{1}_4 = \mathrm{diag}(0,1,1,0)$. The delta function involves only the momentum aligned with the string. The matrix $\bar{1}_4$ instead of the unity matrix appears because the zero mode has only one spin state, there is no summation over two independent spin states as for the free Dirac field. The one-particle state following from it can be defined analogous to the free Dirac field eqs. (B.8), (B.9). The one-particle state is

$$|p,s\rangle = \sqrt{2E_p}a_s^\dagger(p)|0\rangle\,. \tag{B.17}$$

so that

$$\langle ps|qr\rangle = 2E_p(2\pi)\delta^1(p-k)\delta_{rs} \tag{B.18}$$

[7]We introduce M so that in the interaction rate the origin of the different factors can be traced easily.

The renormalization choice eq. (B.17) corresponds to a wave function normalization of $2E_p$ particles per unit length; it reads

$$\langle 0|\psi|ps\rangle = \psi^0_{ps}\mathrm{e}^{-ip\cdot x} = u^0 \frac{M}{\sqrt{\pi}}\mathrm{e}^{-Mr-ip\cdot x}\,, \tag{B.19}$$

where we have defined u^0 through

$$\psi^0 \equiv u^0\alpha = u^0\frac{M}{\sqrt{\pi}}\mathrm{e}^{-Mr} \tag{B.20}$$

The product $u^0\bar{u}^0$, which is needed in the calculation of the squared amplitude $|\mathcal{A}|^2$, in the frame in which the string is aligned with the z-axis and the zero mode has 4-momentum $p^\mu = (p, 0, 0, p)$, is

$$u^0\bar{u}^0 = 2p\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B.21}$$

Similar expressions exist for the anti-particle zero mode.

B.3 Interactions in the string background

The interaction rate for some processes involving a zero mode will be calculated explicitly in the next two appendices. Here we will give some general remarks.

Following [21, 22, 23] interactions in the string background are calculated using the approximation that the incoming and outgoing non-bound states are asymptotically free. That is, we use a free plane wave expansion for these states. The plane wave expansion can then be matched to an angular mode expansion, which is more natural in the string background (as is done in eq. (C.9) below). As the zero mode is bound to the string, it is a good approximation to keep only the the lowest angular modes. Possible amplification factors, due to an amplification of the fermion wave functions of the non-bound states near the string core, will be added by hand, see eq. (D.15) and further.

The main difference between interaction rates involving only asymptotically free states, and interaction rates involving zero modes are the following.

1. The zero mode has only one spin state. This gives factors of order one difference, but does not lead to large quantitative or qualitative differences.

2. Conservation of transverse momentum is violated in the string background. For typical interaction energies $E \lesssim m_\psi$ this has no big consequences.

3. The interaction rate is suppressed by a factor $(E/M)^2$ if the typical interaction energy $E \lesssim M$ with $M \sim m_\psi$ the width of the zero mode. The reason is that the overlap between the zero mode wave function, and the wave functions of the other states is small [22].

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4. Non bound states can have wave functions which are enhanced near the string core compared to the plane wave. This factor is model dependent as it depends on the core structure, the fractional flux and the fermion charges. This can lead to large amplifications of the reaction rate [21, 23].

C. Zero mode decay

To set the ground and to introduce the notation we start with a discussion of the decay rate of a massless, asymptotically free fermion.

C.1 Decay of massless particle

Consider the decay $\psi_R(k,r) \to H(p')\chi_L(k',r')$, mediated by a term in the interaction lagrangian

$$\mathcal{L}_I = hH\bar{\psi}P_L\chi + \text{h.c.} = hH\bar{\psi}_R\chi_L + \text{h.c.} \tag{C.1}$$

Decay is kinematically allowed only if the decay products are massless, and if their momenta are collinear (and aligned with the momentum of the decaying particle ψ_R). The phase space is infinitesimal small. Since all products $p \cdot k$ vanish, with p, k the 4-momenta of the particles involved in the decay process, all Mandelstam variables are zero. The amplitude is

$$\begin{aligned}\mathcal{A} &= \int \mathrm{d}^4x \, \langle k'r'; p'|\mathcal{L}_I|kr\rangle \\ &= (-ih)\int \mathrm{d}^4x \, \bar{u}(k',r')P_L u(k,r)\mathrm{e}^{-i(k-k'-p')} \\ &= (2\pi)^4\delta^4(k-k'-p')(-ih)u(k',r')P_L u(k,r)\,.\end{aligned} \tag{C.2}$$

Here k is the momentum of the initial state fermion, k' that of the final state fermion, and p' that of the final state boson. We use box normalization. The interaction rate per unit volume can than be defined through

$$\dot{P} = \frac{|\mathcal{A}|^2}{VT} \equiv (2\pi)^4\delta^4(k-k'-p')|\mathcal{M}|^2 \tag{C.3}$$

with $\mathcal{A} = (2\pi)^4\delta^4(k-k'-p')i\mathcal{M}$. The decay rate then is

$$\Gamma = \frac{1}{2E_k}\int \mathrm{d}\omega_{k'}\mathrm{d}\omega_{p'}(2\pi)^4\delta^4(k-k'-p')|\mathcal{M}|^2 \tag{C.4}$$

with $\mathrm{d}\omega_p$ the invariant 3-volume

$$\mathrm{d}\omega_p = \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\,. \tag{C.5}$$

Averaging over initial spin states and summing over final states, the matrix element becomes

$$\begin{aligned}|\mathcal{M}|^2 &= \frac{h^2}{2}\sum_{r,r'}\mathrm{Tr}\left[\bar{u}_{k'r'}P_L u_{kr}\bar{u}_{kr}P_R u_{k'r'}\right] \\ &= h^2(k\cdot k') \\ &= 0\end{aligned} \tag{C.6}$$

where the last equality applies for massless collinear particles. Hence, the decay rate for a massless fermion is zero.

C.2 Decay of the zero mode

Consider now decay of the massless zero mode through the same channel eq. (C.1). We have seen that the decay width for a massless free particle is proportional to $k \cdot k'$, which vanishes for massless collinear particles. Processes involving the zero mode do not conserve momentum transverse to the string. This is a consequence of the fact that Lorentz symmetry (translations in the plane transverse to the string) is broken by the presence of the string. Therefore, the decay products of the zero modes can acquire transverse momentum and do not need to be collinear. Can this result in a non-zero decay width?

Let's calculate. The matrix element for $\psi_{\rm zm}(k,r) \to \psi(k',r')$ is

$$\begin{aligned} \langle k'r'|\bar\psi P_L \psi_{\rm zm}|kr\rangle &= \bar u_{k'r'} P_L \psi^0 \mathrm{e}^{-i(k-k')_L x_L} \mathrm{e}^{-ik'_T x_T} \\ &= \bar u_{k's'} P_L u^0_k \mathrm{e}^{-i(k-k')_L x_L} \frac{M}{\sqrt{\pi}} \mathrm{e}^{ik'_T x_T - Mr} \end{aligned} \tag{C.7}$$

where in the last line we have used eq. (B.20). The subscript L and T denote space-time longitudinal (including the time-coordinate) and transverse to the string respectively. For the zero mode $k_T = 0$. The decay amplitude for the process $\psi_{\rm zm}(k) \to H(p')\psi(k',s')$ then is

$$\begin{aligned} \mathcal{A} &= -ih \int \mathrm{d}^4 x \langle k'r'|\bar\psi P_L \psi_{\rm zm}|kr\rangle \langle p'|H|0\rangle \\ &= (2\pi)^2 \delta^2_L(k-k'-p')(-ih) \bar u_{k'r'} P_L u^0_{kr} \frac{g(q_T)}{M} \end{aligned} \tag{C.8}$$

with $q_T = (k'+p')_T$. In the last line we have introduced the dimensionless function g through

$$\begin{aligned} \frac{g(q_T)}{M} &= \int \mathrm{d}^2 x_T \, \frac{M}{\sqrt{\pi}} \mathrm{e}^{ik'_T x_T - Mr} \\ &\approx \frac{M}{\sqrt{\pi}} 2\pi \int \mathrm{d}r r J_0(q_T r) \mathrm{e}^{-Mr} = \frac{2\sqrt{\pi}}{M} \left(1 + \frac{q_T^2}{M^2}\right)^{-3/2} \end{aligned} \tag{C.9}$$

where in the first step we have only kept the lowest angular mode. There is no delta function imposing conservation of transverse momentum. Rather the integration over transverse coordinates lead to a distribution which has maximum $g \sim 2\sqrt{\pi}$ for $q_T = 0$ and decreases (power law) for $q_T \gg M$. There is a non-zero probability for the decay products to acquire transverse momentum.

Using box normalization we can define the interaction rate per unit length as

$$\dot P = \frac{|\mathcal{A}|^2}{LT} = (2\pi)^2 \delta^2_L(k-k'-p') |\mathcal{M}|^2 \,. \tag{C.10}$$

The decay width then is

$$\Gamma = \frac{h^2}{2E_k} \int \mathrm{d}\omega_{k'} \mathrm{d}\omega_{p'} (2\pi)^2 \delta^2_L(k-k'-p') \left(\frac{g(q_T)}{M}\right)^2 \sum_{r'} |\bar u_{k'r'} P_L u^0_k|^2 \,. \tag{C.11}$$

The delta function in the decay width can only be satisfied for massless decay products, with their longitudinal momentum in the same direction as the zero mode momentum. In particular, it implies $E_{k'} = k'$ and $q_T = 0$. Since the decay product are then collinear just as in the decay of a non-bound fermion, we also expect the zero mode decay width to vanish. Indeed, the spinor factor, evaluated in the frame where the string is aligned with the z-axis so that the zero mode momentum is $k^\mu = (k, 0, 0, k)$, is

$$\sum_{r'} |\bar{u}_{k'r'} P_L u^0_{kr}|^2 = \Sigma_{s'} \mathrm{Tr}\left[\bar{u}' P_L u^0 \bar{u}^0 P_R u'\right] = 2k(E_{k'} - k'_z) = 0\,, \qquad \text{(C.12)}$$

where in the second step we have used eq. (B.21), and the equivalent for the massless free spinor $\Sigma_{s'} u' \bar{u}' = \not{k}'$.

Energy-momentum conservation in the longitudinal plane assures that a massless zero mode does not decay.

C.3 Decay of massive bound state

If the zero mode gets an effective mass m no longer $E_k = k$, and thus no longer necessarily $E'_k = k'$, and the decay width can be non-zero. We will assume that the zero mode spinor is unaffected by the effective mass, only the frequency part is altered. The cross section is most easily evaluated in the center of mass frame, where the zero mode is at rest. The spin sum give $\sum_{s'} |\bar{u}_{k's'} P_L u^0_{ks}|^2 \sim m^2$. Introducing an extra delta function the decay width eq. (C.11) then becomes

$$\begin{aligned}
\Gamma &\sim \frac{h^2 m}{M^2} \int \mathrm{d}\omega_{k'} \mathrm{d}\omega_{p'} \mathrm{d}^2 q_T \; (2\pi)^2 \delta^4_L(k + q_T - k' - p') g(q_T)^2 \\
&\sim \frac{h^2 m}{M^2} \int_0^m \mathrm{d}^2 q_T \; g(q_T)^2 \\
&\sim \frac{h^2 m^3}{M^2} \left(1 + \mathcal{O}\left(\frac{m}{M}\right)^2\right).
\end{aligned} \qquad \text{(C.13)}$$

Here, as before, $q_T = (k' + p')_T$ is the transverse momentum provided by the string. The integration over k' and p' can be done in the usual way, and impose energy-momentum conservation in the longitudinal plane. Energy conservation restricts $q_T \lesssim m$. In the last step we have taken the limit $m \ll M \sim m_\psi$, with m_ψ the fermion vacuum mass, to arrive at the final result. For $m \ll M$ there is a wave function suppression. The reason is that then the Compton wavelength m^{-1} is much smaller than the width of the zero mode.

D. Zero mode trapping

We are interested in processes with in the initial state free fermion(s), and in the final state a zero mode. If such reactions are fast compared to the Hubble rate at $T \sim m_\psi$, zero modes are trapped, and chiral currents form.

D.1 Zero mode from string scattering

We first consider the process

$$\psi(k, r) \to \psi_{\mathrm{zm}}(k', r') \qquad \text{(D.1)}$$

in the background of the string. The interaction can take place via either the string Higgs or gauge field. As we will see, the reaction rate is zero as a consequence of energy-momentum conservation.

The amplitude for $\psi(k,r) \to \psi_{\rm zm}(k',r')$ through a fermion-Higgs coupling of the form

$$\mathcal{L} = \lambda\phi\bar{\psi}\psi\,. \tag{D.2}$$

is

$$\mathcal{A} = \int \mathrm{d}^4x \langle S'; k'r'|\mathcal{L}_I|S; kr\rangle = \int \mathrm{d}^4x \langle k'r'|\bar{\psi}\psi|kr\rangle\langle S'|\Phi|S\rangle \tag{D.3}$$

with S denoting the string state. If the back-reaction is small, if the string is not much altered by the scattering, then $S' \approx S$. Here $\Phi = \phi - \phi_0$, the field shifted by the VEV outside the string. We approximate

$$\langle S'|\Phi|S\rangle = \begin{cases} \phi_0, & r < m_\phi^{-1}, \\ 0, & r > m_\phi^{-1}. \end{cases} \tag{D.4}$$

The amplitude then becomes

$$\mathcal{A} \sim (2\pi)^2 \delta_L^2(k-k')(-i\lambda)\bar{u}'^0 u \delta_{rr'} \frac{M\phi_0}{m_\phi^2} \tag{D.5}$$

where we have used that

$$\begin{aligned}\int_0^{m_\phi^{-1}} \mathrm{d}^2x_T \phi_0 \frac{M}{\sqrt{\pi}} \mathrm{e}^{-ik_T\cdot x_t - Mr} &\sim M\phi_0 \int_0^{m_\phi^{-1}} r\mathrm{d}r J_0(k_T r)\mathrm{e}^{-Mr} \\ &\sim \frac{M\phi_0}{m_\phi^2}\left(1 + \mathcal{O}\left(\frac{k_T}{m_\phi}, \frac{M}{m_\phi}\right)\right)\end{aligned} \tag{D.6}$$

The integration over longitudinal momentum gives a delta function enforcing energy-momentum conservation in the longitudinal (t,z) plane. The string is just a stationary background, with no longitudinal string momentum. Note however, that the string can provide transverse momentum so that $q_T \lesssim E$ can be non-zero, with E the typical interaction energy.

Energy momentum conservation in the longitudinal plane, which reads explicitly $E_k = E'_k = k'$ and $k_3 = k'_3 = k'$, is only possible if the incoming fermion is also massless. The same conclusion follows from the Dirac delta in eq. (D.5). The spin of the incoming fermion should be equal to the spin of the zero mode fermion. Only for massless incoming fermions is this possible. The massless spin states outside the string are chiral eigenstates and only in this case is it possible to write the zero mode wave function ψ^0 as a superposition of the fermion states outside the string.[8] We conclude that the process eq. (D.1) to capture zero modes is impossible if the fermions are massive outside the string.

[8]The zero mode spinor is of the form $u^0 = \sqrt{2k}(0\ 1\ 1\ 0)^T$ and the two massless free states are $u_+ = \sqrt{2k}(0\ 0\ 1\ 0)^T$ and $u_- = \sqrt{2k}(0\ 1\ 0\ 0)^T$. It follows $u^0 = u_- + u_+$.

We will show here that the cross section is also zero for massless incoming fermions. The cross section per unit length is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}l} \sim \frac{\lambda^2}{E} \int \frac{\mathrm{d}k'}{(2\pi)2k'} \delta_L^2(k-k') \frac{1}{2} \delta_{rr'} |\bar{u'}^0 u|^2 \left(\frac{M\phi_0}{m_\phi^2}\right)^2 . \tag{D.7}$$

Evaluating the spin term in the frame where the zero mode momentum is $k'^\mu = (k', 0, 0, k')$ gives

$$\begin{aligned} \delta_L^2(k-k')\delta_{rr'}|\bar{u}^0(k')u(k,s)|^2 &= \delta_L^2(k-k')2k'(E_k - k_3) \\ &= 0 \end{aligned} \tag{D.8}$$

String scattering through the interaction term $\mathcal{L} = iq\bar{\psi}\gamma^\theta A_\theta \psi$ gives similar expression. It can easily be checked that also in this case the cross section is zero.

Can string scattering lead to zero mode capture if the zero mode is effectively massive? Energy momentum conservation in the longitudinal plane gives $k_T^2 + m_\psi^2 = m^2$, with k_T the transverse momentum of the incoming particle, m_ψ the vacuum mass, and m the effective zero mode mass. It follows that zero mode capture through eq. (D.1) is impossible for

$$m_\psi > m \tag{D.9}$$

which is the case of interest.

D.2 Four fermion scattering

Consider

$$\psi(k,r) + \bar{\psi}(p,s) \to \psi_{\mathrm{zm}}(k') + \bar{\psi}(p',s') \tag{D.10}$$

mediated by either a Higgs or gauge boson. Since the zero mode wave function For simplicity we consider only s-channel Higgs mediation explicitly; other diagrams and interference terms give similar contributions. The amplitude for this process is

$$\begin{aligned} \mathcal{A} &= (-i\lambda)^2 \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \bar{v}(p,s)u(k,r)\frac{1}{q^2+m_\phi^2}\bar{u}^0(k')v(p',s')\mathrm{e}^{-i(k+p-q)\cdot x_1 - i(q-k'-p')\cdot x_2 - Mr} \\ &\sim (2\pi)^2\delta_L^2(k+p-k'-p')\frac{(-i\lambda)^2}{m_\phi^2}\bar{v}(p,s)u(k,r)\bar{u}^0(k')v(p',s')\frac{g((q-p')_T)}{M} \end{aligned} \tag{D.11}$$

where we have assumed that the interaction takes place a distance $r < m_f^{-1}$ away from the string core. In the opposite limit $r \ll m_f^{-1}$ the zero mode wave function approaches zero exponentially fast, and the cross section is zero. In the second line we have specialized to interaction energies $q = k + p \ll m_\phi$. The function g is defined in eq. (C.9). The cross section (for interactions taking place within a distance $r < m_f^{-1}$ from the string center, is

$$\sigma \sim \frac{\lambda^4}{m_\phi^4} \int \mathrm{d}\omega_{p'} \mathrm{d}\omega_{k'} \mathrm{d}^2 q_T \; \delta_L^4(q-k'-p') \frac{1}{4} \Sigma_{r,s,s'} |\bar{v}u\bar{u}^{0'}v'|^2 \frac{g(q_T)^2}{M^2} \tag{D.12}$$

with $d\omega_{p'}$ given in eq. (C.5), and $d\omega_{k'} = dk/(4\pi k)$. The spin sum is evaluated in the frame in which the zero-mode momentum is $k'^{\mu} = (k', 0, 0, k')$:

$$\frac{1}{4}\Sigma_{r,s,s'}|\bar{v}u\bar{u}^{0'}v'|^2 = 4k'\left((k_0 - k_3)(m^2 - p\cdot p') - (p'_0 - p'_3)(m^2 - k\cdot p)\right) \tag{D.13}$$

which is non-zero for massive fermions (so that $|k_0| \neq |k_3|$ and $|p'_0| \neq |p'_3|$). For interaction energies $q \sim m_\psi$ the cross section then is

$$\sigma \sim \lambda^4 \frac{m_\psi^4}{m_\phi^4 M^2} \int d^2 q_T g^2 \overset{M\sim m_\psi}{\longrightarrow} \lambda^4 \frac{m_\psi^2}{m_\phi^4} \tag{D.14}$$

For $m_\psi \ll M$ there is a $(m_\psi/M)^2$ suppression as the Compton wavelength m_ψ^{-1} is much smaller than the width of the zero mode. In this calculation we have use that the free particles are of the plane wave form. However their wave function may be amplified on the scale of the typical interaction length m_ψ^{-1}. Defining

$$A = \frac{|\tilde{u}(r = m_\psi^{-1})|}{|u(r = m_\psi^{-1})|} \tag{D.15}$$

with $\tilde{u}$ the wave function in the string background and u, as before, the plane wave. The cross section corrected by these amplification factors becomes

$$\sigma_{\text{amp}} \sim A^6 \lambda^4 \frac{m_\psi^2}{m_\phi^4}\,. \tag{D.16}$$

The amplification factor depends on the core profile and on the fractional flux. We expect the amplification factor to be

$$A \sim \left(\frac{1}{kr}\right)^p \lesssim \left(\frac{m_\psi}{k}\right) \sim 1 \tag{D.17}$$

with the coefficient $p \leq 1$.[9]

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